A POINT SOURCE OF LUMINOUS RADIATION IN A SCATTERING MEDIUM

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This paper establishes the distribution of luminous radiation from a point source located in a scattering medium. It is assumed that the scattering is isotropic: that is that any element of the medium emits radiation in all directions with equal intensity. This problem was first considered in a paper by Ambartsumian [1]. Some problems of a similar type have been considered in connection with the diffusion of neutrons (see, for example, [2]).

1. A number of quantities are determined in this article which have interest in connection with radiation from a point source. Thus a quantity B exists, equal to the radiation absorbed in unit volume, and also the intensity of radiation arriving at a given point from a given direction, which permits the determination of the intensity of the halo around a point source. Further, the total quantity of luminous radiation falling on the unit area from the source, is determined.

We first give the solution of the auxiliary one-dimensional problem (depending on the coordinate z) of the distribution in space of radiation from a specified layer, which consists of point sources, and which coincides with the plane xoy. In this connection it is assumed that the layer is transparent in relation to radiation being propagated from one half-space into the other. The original integral equation can be obtained on the basis of the equation of ray propagation (see [3]). In the present case, however, in view of the fact that the radiation is isotropic, the equation can be derived by a simpler method.

Suppose a certain elementary volume is the source of illumination in which the emission is isotropic; that is, with equal intensity in all directions. In such case, the intensity of the radiation at a distance R from the source will be [3]

$$J = \frac{\varepsilon}{4\pi R^2} e^{-kR} \tag{1}$$

Here k is the coefficient of absorption characteristic of the given medium. The quantity ϵ characterizes the strength of the source; it may be measured, for example, in heat units.

The quantity of radiation absorbed by an elementary volume dV, as can easily be shown (see [3]), will be

$$q = kJ\,dV \tag{2}$$

Here J is the intensity of the radiation falling on the given element in a given direction.

The total quantity of radiation absorbed in this case will be

$$B\,dV = k\iint J d\Omega\,dV \tag{3}$$

Here the integration is carried out over the whole solid angle. Since the volume absorbing radiation is itself a source, that is radiates some part of whatever reaches it, the intensity of the illumination from the element will be

$$J_1 = \frac{\sigma B}{4\pi R^2} dV \tag{4}$$

Here σ is the ratio of the quantity of light scattered to that absorbed.

On the basis of the relations (1), (2), (3), and (4), the fundamental integral equation can be derived for the current function B.

We define the radiation which comes from a layer with coordinate ζ and is absorbed by the element A, which is located for convenience at the origin of coordinates by:

$$dB(z) = \sigma k B(\zeta)\,d\zeta \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\exp\left[k\sqrt{x^2+y^2+(z-\zeta)^2}\right]}{4\pi\left(x^2+y^2+(z-\zeta)^2\right)}\,dx\,dy \tag{5}$$

We transform the double integral introduced in equation (5). For this purpose, we introduce the following substitutions

$$z-\zeta=\xi, \quad \sqrt{x^2+y^2}=r, \quad \sqrt{r^2+\xi^2}=s, \quad ks=\vartheta, \quad kz=\tau, \quad k\zeta=\omega$$

In this case

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\exp\left[-k\sqrt{x^2+y^2+(z-\zeta)^2}\right]}{4\pi\left(x^2+y^2+(z-\zeta)^2\right)}\,dx\,dy = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\exp\left[-k\sqrt{x^2+y^2+\xi^2}\right]}{4\pi\left(x^2+y^2+\xi^2\right)}\,dx\,dy =$$

$$=\int_0^\pi\int_0^\infty \frac{\exp\left[-k\sqrt{r^2+\xi^2}\right] r\,dr\,d\theta}{4\pi(r^2+\xi^2)} = \frac{1}{2}\int_0^\infty \frac{\exp\left[-k\sqrt{r^2+\xi^2}\right]}{\sqrt{r^2+\xi^2}}\frac{r\,dr}{\sqrt{r^2+\xi^2}} =$$

$$=\frac{1}{2}\int_\xi^\infty \frac{e^{-ks}\,ds}{s} = \frac{1}{2}\int_{k\xi}^\infty \frac{e^{-\vartheta}}{\vartheta}\,d\vartheta = -\frac{1}{2}\,\mathrm{Ei}\,(-\,|k\xi|) =$$

$$= -\frac{1}{2}\,\mathrm{Ei}\,(-k\,|z-\xi|) = -\frac{1}{2}\,\mathrm{Ei}\,(-\,|\tau-\omega|) \tag{6}$$

Here Ei (α) is the integral-exponential function determined by the following expression

$$\mathrm{Ei}\,(\alpha) = \int_{-\infty}^{\alpha} \frac{e^{-t}}{t}\,dt$$

On the basis of this relation, equation (5) takes the form

$$dB\,(z) = -\frac{\sigma}{2}\,k d\zeta\,\mathrm{Ei}\,(-\,|\tau-\omega|) = -\frac{\sigma}{2}\,d\omega\,\mathrm{Ei}\,(-\,|\tau-\omega|) \tag{7}$$

Proceeding in an analogous manner, we get the radiation from the layer of point sources located in the plane xoy

$$B_0\,(z) = -\,\varepsilon\,\frac{k}{2}\,\mathrm{Ei}\,(-\,|kz|) \tag{8}$$

Here ϵ is a coefficient related to the density of the distributed sources.

Since the element A receives radiation from all layers of material in space, and also from the layer of sources distributed on the plane xoy, the integral equation for determining $B(z)$ will have the following form:

$$B\,(\tau) = -\,\varepsilon\,\frac{k}{2}\,\mathrm{Ei}\,(-\,|\tau|) - \frac{\sigma}{2}\int_{-\infty}^{\infty} B\,(\omega)\,\mathrm{Ei}\,(-\,|\tau-\omega|)\,d\omega \tag{9}$$

2. We develop the solution of equation (9). For this purpose we introduce into consideration the functions

$$G\,(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{Ei}\,(-\,|\tau|)\,e^{i\tau u}\,d\tau \tag{10}$$

$$F\,(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} B\,(\tau)\,e^{i\tau u}\,d\tau \tag{11}$$

Multiplying both parts of equation (9) by $e^{i\tau u}$ and integrating in the limits from $-\infty$ to $+\infty$, we obtain

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[-\varepsilon \frac{k}{2} \operatorname{Ei}(-|\tau|) - \frac{\sigma}{2} \int_{-\infty}^{\infty} B(\tau) \operatorname{Ei}(-|\tau-\omega|)\, d\omega \right] e^{i\tau u}\, d\tau$$

Introducing the variable t, such that $\tau = t + \omega$ we obtain

$$F(u) = -\varepsilon \frac{k}{2} G(u) - \frac{\sigma}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} B(\omega) e^{iu\omega}\, d\omega \int_{-\infty}^{\infty} \operatorname{Ei}(-|t|)\, e^{-tu}\, dt$$

or

$$F(u) = -\varepsilon \frac{k}{2} G(u) - \frac{\sigma}{2} \sqrt{2\pi}\, G(u)\, F(u)$$

From this, finally, we obtain

$$F(u) = \frac{-\,{}^1\!/_2\, \varepsilon k\, G(u)}{1 + {}^1\!/_2\, \sigma \sqrt{2\pi}\, G(u)} \tag{12}$$

We determine the function $G(u)$. Taking advantage of the symmetry properties of the integrand, we have

$$G(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \operatorname{Ei}(-|\tau|)(\cos \tau u + i \sin \tau u)\, d\tau =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \operatorname{Ei}(-|\tau|) \cos \tau u\, d\tau = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \operatorname{Ei}(-|\tau|) \cos \tau u\, d\tau =$$

$$= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \operatorname{Ei}(-\tau) \cos \tau u\, d\tau \tag{13}$$

Or finally, after integrating by parts,

$$G(u) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \operatorname{Ei}(-|\tau|)\, e^{i\tau u}\, d\tau = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \operatorname{Ei}(-\tau) \cos \tau u\, d\tau =$$

$$= -\sqrt{\frac{2}{\pi}}\, \frac{1}{u} \operatorname{arc\,tg} u \tag{14}$$

Using the inverse Fourier transformation, and allowing for the symmetry properties of the integrand, for the function $B(\tau)$ we obtain

$$B(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(u)\, e^{i\tau u}\, du = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} F(u) \cos \tau u\, du$$

Using euqations (12) and (14) for $F(u)$ and $G(u)$, and setting $\tau = kz$, we finally obtain

$$B(\tau) = \varepsilon \frac{k}{2} \frac{1}{\pi} \int_{0}^{\infty} \frac{u^{-1} \operatorname{arc\,tg} u}{1 - \sigma u^{-1} \operatorname{arc\,tg} u} \cos kzu\, du \tag{15}$$

3. We now set up the relation between the solution for the one-dimensional problem, and the spherically symmetrical problem. For the case of a point source, let the quantity of radiation absorbed per unit volume, at a distance r from the source, be represented by the function $\phi(z)$. In this case the function $B(z)$, giving the same quantity for the one-dimensional problem, may be used by means of a summation of the solutions corresponding to different sources; that is, by means of an integration of the spherically symmetrical solution

$$B(z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi\left(\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}\right)d\xi\,d\eta=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi\left(\sqrt{\xi^2+\eta^2+z^2}\right)d\xi\,d\eta=\int_0^{2\pi}\int_0^{\infty}\varphi\left(\sqrt{z^2+\rho^2}\right)\rho d\rho\,d\theta=$$

$$=2\pi\int_0^{\infty}\varphi\left(\sqrt{z^2+\rho^2}\right)\rho d\rho$$

Or, setting $\sqrt{z^2+\rho^2}=\zeta$, we obtain

$$B(z)=2\pi\int_z^{\infty}\varphi(\zeta)\,\zeta\,d\zeta \tag{16}$$

From this, we get

$$\varphi(r)=-\frac{1}{2\pi}\,\frac{B'(r)}{r} \tag{17}$$

Now if we take advantage of expression (15), obtained for the one-dimensional case, the formula based on (17) for radiation absorbed in the case of a point source is

$$\varphi(r)=-\frac{1}{2\pi}\,\frac{1}{r}\,\frac{d}{dr}\left\{k\varepsilon\,\frac{1}{\pi}\int_0^{\infty}\frac{u^{-1}\operatorname{arc\,tg}u}{1-\sigma u^{-1}\operatorname{arc\,tg}u}\cos kru\,du\right\} \tag{18}$$

We transform this expression. Using the inverse Fourier transformation as applied to (14), we obtain

$$\frac{2}{\pi}\int_0^{\infty}\left(\frac{1}{u}\operatorname{arc\,tg}u\right)\cos kru=-\operatorname{Ei}(-|kr|) \tag{19}$$

In this case

$$\varphi_0(r)=-\frac{1}{r}\,\frac{d}{dr}\left\{k\varepsilon\,\frac{1}{4\pi}\int_0^{\infty}\frac{1}{u}\operatorname{arc\,tg}u\cos kru\,du\right\}= \tag{20}$$

$$=-\frac{k\varepsilon}{4\pi r}\,\frac{d}{dr}\left[-\operatorname{Ei}(-kr)\right]=-\frac{k\varepsilon}{4\pi r}\,\frac{d}{dr}\left[\int_{kr}^{\infty}\frac{e^{-s}}{s}\,ds\right]=\varepsilon k\,\frac{e^{-kr}}{4\pi r^2}$$

Equation (18) for $\phi(r)$ may be transformed in the following manner:

$$\varphi(r) = -\frac{1}{r}\frac{d}{dr}\frac{k\varepsilon}{2\pi^2}\left\{\int_0^\infty \left(\frac{1}{u}\operatorname{arc\,tg} u\right)\cos kru\,du + \int_0^\infty \left[\frac{\sigma u^{-2}\operatorname{arc\,tg}^2 u}{1-\sigma u^{-1}\operatorname{arc\,tg} u}\right]\cos kru\,du\right\}$$

As a result, allowing for (20), we obtain

$$\varphi(r) = k\frac{\varepsilon e^{-kr}}{4\pi r^2} - \frac{1}{r}\frac{d}{dr}\left[\frac{k\varepsilon}{2\pi^2}\int_0^\infty \frac{\sigma u^{-2}\operatorname{arc\,tg}^2 u}{1-\sigma u^{-1}\operatorname{arc\,tg} u}\cos kru\,du\right] =$$

$$= k\frac{\varepsilon e^{-kr}}{4\pi r^2} + \frac{\varepsilon k^2}{2\pi^2}\frac{1}{r}\int_0^\infty \frac{\sigma u^{-1}\operatorname{arc\,tg}^2 u}{1-\sigma u^{-1}\operatorname{arc\,tg} u}\sin kru\,du \qquad (21)$$

In this formula ϵ is the strength of the source of radiation, which can be measured, for example, in calories per second.

The form of this expression makes it possible to establish the effect of scattering. The first term in (21) corresponds to absorption where scattering is absent, and the second term allows for its effect.

4. We derive the intensity of the halo formed around a point source. We consider point A located at a distance R from the source. We construct a cone with solid angle $d\,\Omega$. Its vertex is at point A, and its axis, with the radius from the source to point A, forms an angle equal to a.

We isolate an element located at a distance ρ from point A, the volume of which is equal to $\rho^2 d\rho\, d\Omega$. The distance of this element from the source of radiation is designated by r. In this case, the radiation reaching point A from a direction forming an angle a with the radius is determined in the following form:

$$J(R,\alpha)\,d\Omega = \int_0^\infty \sigma\varphi(r)\frac{e^{-k\rho}}{4\pi\rho^2}\rho^2 d\rho\,d\Omega$$

Since $r = \sqrt{\rho^2 - 2\rho R\cos a + R^2}$, the expression for $J(R, a)$ can be expressed in the form

$$J(R,\alpha) = \frac{\sigma}{4\pi}\int_0^\infty \varphi\left(\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)e^{-k\rho}\,d\rho \qquad (22)$$

Substituting (21) in this expression, we obtain

$$J(R,\alpha) = \varepsilon\frac{\sigma k}{4\pi}\int_0^\infty \frac{\exp\left(-k\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)}{\rho^2 - 2\cos\alpha\rho R + R^2}e^{-k\rho}\,d\rho +$$

$$+\varepsilon\frac{\sigma k^2}{2\pi^2}\int_0^\infty \frac{\sigma u^{-1}\operatorname{arc\,tg}^2 u}{1-\sigma u^{-1}\operatorname{arc\,tg} u}\,du\int_0^\infty \frac{\sin\left(ku\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)}{\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}}e^{-k\rho}\,d\rho \qquad (23)$$

We note that the first of these integrals can be expressed through an integral-exponential function with a complex argument.

The current $q(R)$ falling from the direction of the source on an area normal to the radius is made up of radiation coming directly from the source and of radiation coming from the halo. In order to find the secondary component, it is necessary to integrate the current falling on the area from different directions.

On the basis of (1) and (23) for this quantity, we obtain the following expression

$$q(R) = \frac{\varepsilon e^{-kR}}{4\pi R^2} + \int_0^{1/2\pi} \pi \sin 2\alpha\, d\alpha \left[\frac{\sigma\varepsilon k}{4\pi}\int_0^\infty \frac{\exp\left(-k\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)}{\rho^2 - 2\cos\alpha\rho R + R^2} e^{-k\rho}\, d\rho + \right.$$
$$\left. + \frac{\sigma\varepsilon k^2}{2\pi^2}\int_0^\infty \frac{\sigma u^{-1}\,\mathrm{arc\,tg}^2 u}{1 - \sigma u^{-1}\,\mathrm{arc\,tg}\, u}\, du \left\{\int_0^\infty \frac{\sin\left(ku\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)}{\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}}\, e^{-k\rho}\, d\rho\right\}\right]$$

This expression can be transformed as follows:

$$q(R) = \frac{\varepsilon e^{-kR}}{4\pi R^2} + \varepsilon\frac{\sigma k}{4}\int_0^{1/2\pi} \sin 2\alpha\, d\alpha \left[\int_0^\infty \frac{\exp\left(-k\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)}{\rho^2 - 2\cos\alpha\rho R + R^2} e^{-k\rho} d\rho\right] +$$
$$+ \varepsilon\frac{\sigma^2 k^2}{2\pi}\int_0^{1/2\pi} \sin 2\alpha\, d\alpha \left[\int_0^\infty \frac{u^{-1}\,\mathrm{arc\,tg}^2 u}{1 - \sigma u^{-1}\,\mathrm{arc\,tg}\, u}\, du \int_0^\infty \frac{\sin\left(ku\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}\right)}{\sqrt{\rho^2 - 2\cos\alpha\rho R + R^2}} e^{-k\rho} d\rho\right] \tag{24}$$

We introduce the dimensionless quantities $\theta = kR$ and $\eta = \rho/R$. Thereupon (24) can be transformed into the final form:

$$q(R) = \varepsilon\frac{e^{-\theta}}{4\pi R^2}\left\{1 + 2\pi\sigma\theta e^{\theta}\int_0^{1/2\pi} \sin\alpha\cos\alpha\, d\alpha \left[\int_0^\infty \frac{e^{-\theta\sqrt{\eta^2 - 2\cos\alpha\eta + 1}}}{\eta^2 - 2\cos\alpha\eta + 1} e^{-\theta\eta} d\eta\right] + \right.$$
$$\left. + 4\sigma^2\theta^2 e^{\theta}\int_0^{1/2\pi} \sin\alpha\cos\alpha\, d\alpha \left[\int_0^\infty \frac{\frac{1}{u}\,\mathrm{arc\,tg}^2 u\, du}{1 - \frac{\sigma}{u}\,\mathrm{arc\,tg}\, u}\int_0^\infty \frac{\sin\left(u\theta\sqrt{\eta^2 - 2\cos\alpha\eta + 1}\right)}{\sqrt{\eta^2 - 2\cos\alpha\eta + 1}} e^{-\theta\eta} d\eta\right]\right\} \tag{25}$$

When scattering is absent, then $\sigma = 0$, and we obtain the elementary formula. In an approximate computation of scattering, where the coefficient σ is small, the only significant terms are the first two in the brackets.

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